J. Fluid Mech. (1972), *vol.* 54, *part* 3, *pp.* 393–415
Printed in Great Britain

Non-local effects in the stability of flow between eccentric rotating cylinders

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(Received 5 January 1972 and in revised form 28 April 1972)

In this paper the linear stability of the flow between two long eccentric rotating circular cylinders is considered. The problem, which is of interest in lubrication technology, is an extension of the classical Taylor problem for concentric cylinders. The basic flow has components in the radial and azimuthal directions and depends on both of these co-ordinates. As a consequence the linearized stability equations are *partial differential equations* rather than ordinary differential equations. Thus standard methods of stability theory are not immediately useful. However, there are two small parameters in the problem, namely δ, the clearance ratio, and ϵ, the eccentricity. By letting these parameters tend to zero in such a way that $\delta^{\frac{1}{2}}$ is proportional to ϵ, a global solution to the stability problem is obtained without recourse to the concept of 'local instability', or 'parallel-flow' approximation, so commonly used in boundary-layer stability theory. It is found that the predictions of the present theory are at variance with what is given by a 'local' theory. First, the Taylor-vortex amplitude is found to be largest at about 90° downstream of the region of 'maximum local instability'. This result is given support by some experimental observations of Vohr (1968) with $\delta = 0{\cdot}1$ and $\epsilon = 0{\cdot}475$, which yield a corresponding angle of about 50°. Second, the critical Taylor number rises with ϵ, rather than initially decreasing with ϵ as predicted by local stability theory using the criteria of maximum local instability. The present prediction of the critical Taylor number agrees well with Vohr's experiments for ϵ up to about 0·5 when $\delta = 0{\cdot}01$ and for ϵ up to about 0·2 when $\delta = 0{\cdot}1$.

1. Introduction

There is an interest in lubrication technology in the phenomenon of Taylor-vortex instability in a journal bearing. As a model we consider the flow between two long rotating circular cylinders (radii a and b with $b > a$) when their axes are not coincident. In lubrication problems the mean gap between the cylinders $(b-a)$ is very small $(b-a \ll a)$, and the distance between the cylinder axes (ae) can be a substantial fraction of that mean gap. The eccentricity parameter ϵ is defined by $ae = \epsilon(b-a)$, so that $0 \leqslant \epsilon < 1$.

Experiments on the instability of the flow between eccentric cylinders have been made by a number of workers, including Cole (1957, 1965), Kamal (1966), Vohr (1967, 1968), Castle & Mobbs (1968), Versteegen & Jankowski (1969), Coney & Mobbs (1970), Castle, Mobbs & Markho (1971) and Frêne & Godet (1971). There are two features of the observations which we wish to explain in this paper. First, the critical Taylor number (for the occurrence of the instability) varies with the eccentricity parameter. Second, Vohr reports that, in one experiment at least (with $\epsilon = 0{\cdot}475$, $b-a = 0{\cdot}099a$), the Taylor-vortex secondary motion appeared to have its maximum 'activity' at an azimuthal position some 50° downstream from the position of maximum gap when the inner cylinder was rotating and the outer was at rest. This latter result is rather surprising at first sight, since the 'most unstable' zone (DiPrima 1963) appears to be that where the gap between the cylinders is greatest. The mathematical theory suggests almost immediately a possible explanation for the latter phenomenon, in the following way. Since the basic flow between the cylinders depends strongly upon the azimuthal angle, the linearized equations for the instability of the flow are *partial* differential equations in the radial and azimuthal co-ordinates. Boundary conditions of no-slip are needed at the cylinders and the solution must be single-valued. Thus the solution required is a 'global' one, in that the flow field at all points must affect the stability characteristics and produce, in some way, the observed position of maximum Taylor-vortex activity.

A similar problem has long been embedded in the literature, namely the theory of boundary-layer instability. There also, the basic flow depends on two co-ordinates, so that the true stability equations are partial differential equations. Moreover, in that case it has been customary to approximate the partial differential equations for stability by an ordinary differential equation, namely the Orr–Sommerfield equation with the local boundary-layer velocity profile. This is known as the parallel-flow approximation. To the author's knowledge, no one has succeeded in developing a self-consistent theory which avoids the parallel-flow approximation and takes account properly of the effects on stability of the velocity-profile variation in the flow direction; such effects remain an enigma. In our present problem, however, the requirement of single valuedness enables the difficulty to be overcome, so that a *local* theory (as implied by the parallel-flow approximation) can be replaced by a truly global one. We believe that in addition to its contribution in lubrication technology, the present analysis is an example from the interesting class of mathematical stability problems in which the basic flow varies significantly in two co-ordinates. However, the implications for problems of boundary-layer instability remain to be assessed.

The procedure to be adopted in the present paper is the following. The basic flow is calculated in § 2 from the equations of motion, written in a modified bipolar co-ordinate system (DiPrima & Stuart 1972); an expansion is made in two small parameters, a modified Reynolds number R_M and a parameter α, representing effects of surface curvature, which is small with reference to the gap between the cylinders. The zeroth approximation is exactly that which is given by the solution of the approximate 'Reynolds' equation of classical lubrication theory, while terms of order R_M and α give first-order inertial and curvature corrections. For reasons which are clarified in § 4, terms of higher order are not needed for the purposes of the present paper. Thus, with the basic flow known to order α and R_M, the partial differential equations of stability (against perturbations periodic along the axis) are formulated in §3. An asymptotic approximation relating α, R_M, ϵ and a Taylor number T is then discussed and explained in §4. This asymptotic expansion results in a set of simpler (and *ordinary*) differential equations for stability, but one which still retains, as a set, a global property. Section 5 is devoted to the solution of these equations by what is essentially the method of multiple scales. This is followed in §6 by a comparison of the analytical results with the experimental evidence of Vohr (1968) and others. A discussion of the present work and of its implications is given in §7. Finally we note that the reader may wish to consult a study of an MHD stability problem (Baldwin 1972), where similar mathematical ideas are used.

2. The basic laminar flow

In our study of the stability of flow between two eccentric rotating cylinders, when the gap between them is very small, it is not sufficiently accurate to use the flow field given by the solution of the Reynolds lubrication equation, and we find it convenient to consider the equations of two-dimensional viscous flow in the modified bipolar system of co-ordinates, as used by Wood (1957). The analysis has been carried out by DiPrima & Stuart (1972) to the accuracy needed here. We shall briefly outline the procedure and summarize the results in this section. The inner and outer cylinders have radii a and b, with linear speeds q_1 and q_2 measured in the anti-clockwise direction. The centres of the cylinders are set at a distance ae apart (figure 1), where

$$e = \epsilon\delta, \quad \delta = (b-a)/a \tag{2.1}$$

and

$$0 \leqslant \epsilon < 1, \tag{2.2}$$

the latter condition ensuring that the two cylinders do not touch. In lubrication theory, ϵ is known as the eccentricity and δ as the clearance ratio. The r, θ polar co-ordinate system shown in figure 1 has its origin at the axis of the inner cylinder, with the ray $\theta = 0$ passing through the axis of the outer cylinder.

We follow Wood and use the conformal transformation

$$z = \frac{a(\zeta+\gamma)}{1+\gamma\zeta}, \quad z = re^{i\theta}, \quad \zeta = \rho e^{i\phi}, \tag{2.3}$$

where

$$\gamma = \frac{-(1+\beta)+[(1+\beta)^2-4\epsilon^2\beta]^{\frac{1}{2}}}{2\epsilon\beta}, \tag{2.4}$$

$$\beta = \frac{1+\delta+\epsilon\delta-\gamma}{1-(1+\delta)\gamma-\epsilon\delta\gamma}. \tag{2.5}$$

The co-ordinate curves ρ = constant are circles and, in particular, the inner and outer cylinders are given respectively by $\rho = 1$ and β. The ray $\phi = 0$ coincides with $\theta = 0$. In the limit $\epsilon \to 0$ the ρ, ϕ co-ordinates become identical with the r, θ co-ordinates, which is an advantage of this co-ordinate system compared with

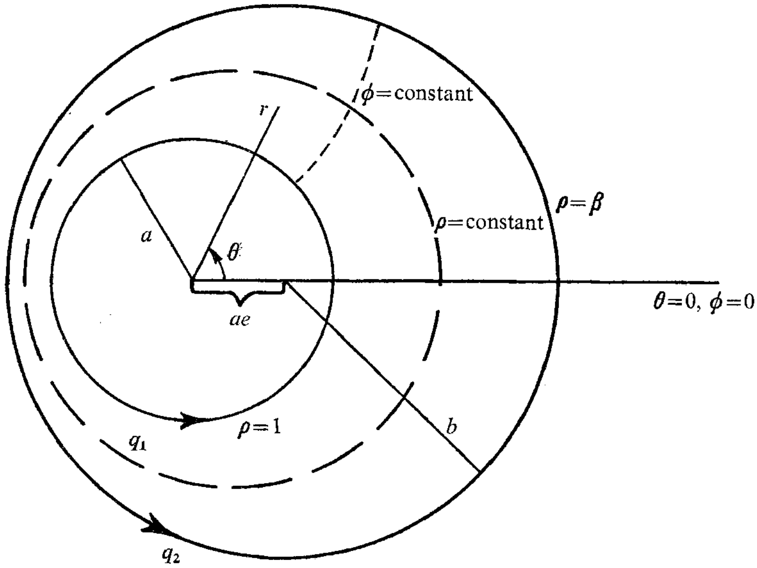

FIGURE 1. Geometry and co-ordinate systems.

the usual bi-polar co-ordinate system. The Jacobian J of the transformation (2.3) is given by

$$J = (1+2\gamma\rho\cos\phi+\gamma^2\rho^2)^2/(1-\gamma^2)^2, \tag{2.6}$$

and the length element in two dimensions is given by

$$ds^2 = dr^2+r^2\,d\theta^2 = \frac{a^2}{J}\,d\rho^2+\frac{a^2\rho^2}{J}\,d\phi^2. \tag{2.7}$$

Following DiPrima & Stuart (1972), we define

$$u_\rho = \frac{\alpha q_1 J^{\frac{1}{2}}}{\rho}\frac{\partial\Psi}{\partial\phi}, \quad u_\phi = -q_1 J^{\frac{1}{2}}\frac{\partial\Psi}{\partial x}, \tag{2.8}$$

$$\rho = 1+\alpha(x+\tfrac{1}{2}), \quad \alpha = \beta-1, \tag{2.9}$$

where u_ρ and u_ϕ are the dimensionless components of velocity in the directions of ρ and ϕ increasing, respectively. Then the equation for the dimensionless axial vorticity Ω is

$$\frac{R_M}{\rho}\left(\frac{\partial\Psi}{\partial\phi}\frac{\partial\Omega}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Omega}{\partial\phi}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\alpha}{\rho}\frac{\partial}{\partial x}+\left(\frac{\alpha}{\rho}\right)^2\frac{\partial^2}{\partial\phi^2}\right)\Omega, \tag{2.10}$$

where

$$\Omega = -J\left(\frac{\partial^2}{\partial x^2}+\frac{\alpha}{\rho}\frac{\partial}{\partial x}+\left(\frac{\alpha}{\rho}\right)^2\frac{\partial^2}{\partial\phi^2}\right)\Psi. \tag{2.11}$$

The parameter R_M, the modified Reynolds number, is given by

$$R_M = (q_1 a/\nu)\,\alpha^2. \tag{2.12}$$

The boundary conditions are

$$\left.\begin{aligned}\frac{\partial\Psi}{\partial x} &= -J^{-\frac{1}{2}}, \quad \frac{\partial\Psi}{\partial\phi} = 0 \quad \text{at} \quad x = -\tfrac{1}{2},\\ \frac{\partial\Psi}{\partial x} &= -\mu J^{-\frac{1}{2}}, \quad \frac{\partial\Psi}{\partial\phi} = 0 \quad \text{at} \quad x = \tfrac{1}{2},\end{aligned}\right\} \tag{2.13}$$

μ denoting q_2/q_1. [We note that μ is denoted by η in DiPrima & Stuart (1972).] In addition the pressure, which is given by integration of the momentum equations associated with (2.10), is required to be a single-valued function of ϕ. This condition, together with (2.10), (2.11) and (2.13), can be used to obtain Ψ as a function of x and ϕ.

Since we are interested in the application to lubrication, the parameter δ of (2.1) is small; moreover, it can be shown from (2.4) and (2.5) that

$$\alpha(\delta,\epsilon) = \beta - 1 = \delta(1-\epsilon^2)^{\frac{1}{2}}\{1-\tfrac{1}{2}\delta[1-(1-\epsilon^2)^{\frac{1}{2}}]\}+O(\delta^3), \tag{2.14}$$

so that α is small also. As for R_M, we shall see later that it, too, is to be small. Thus we expand Ψ as a power series in α and R_M:

$$\Psi = \Psi_{00}(x,\phi;\,\epsilon,\mu)+R_M\Psi_{10}(x,\phi;\epsilon,\mu)+\delta\alpha\Psi_{01}(x,\phi;\epsilon;\mu) \\ +R_M^2\Psi_{20}(x,\phi;\,\epsilon,\mu)+O(\alpha^2,\alpha R_M,R_M^3). \tag{2.15}$$

From this formula, together with the associated series

$$J(x,\phi;\epsilon,\alpha) = J_0(\phi;\,\epsilon)+\alpha J_1(x,\phi,\epsilon)+O(\alpha^2), \tag{2.16}$$

it is possible to calculate u_ρ and u_ϕ from (2.8) to $O(\alpha)$ and $O(R_M)$.

Details and results of the calculation of Ψ_{00}, Ψ_{10}, Ψ_{01}, J_0 and J_1, together with associated formulae for the pressure distribution, can be found in the paper by DiPrima & Stuart (1972). We note here that Ψ_{00} yields the Sommerfeld pressure distribution, torque and load, that $\alpha\Psi_{01}$ alters these quantitatively by a small amount, but that $R_M\,\Psi_{10}$ has the more significant property of rotating the load vector. We note also that there is 'separation' (with a region of reversed flow) for

$$\epsilon \geqslant 0{\cdot}30278+0{\cdot}03818\delta, \tag{2.17}$$

a result which is not affected by Ψ_{10}. The term Ψ_{20} has been retained in (2.15) because our stability calculations require $R_M = O(\alpha^{\frac{1}{2}})$. However, Ψ_{20} in fact is proportional to ϵ, so that $R_M^2\,\psi_{20}$ is of order $\alpha\epsilon$, a term which we shall see to be negligible in §4. Thus Ψ_{20} has not been calculated in detail.

3. The equations of stability for the flow

We now need to consider the momentum equations of viscous flow in the dimensionless co-ordinate system (ρ,ϕ,ζ), where ζ, the axial co-ordinate, should not be confused with the complex number of the conformal transformation (2.3). (The axial distance is actually $a\zeta$.) The momentum equations and continuity equation are

$$\frac{du_\rho}{dt}-\frac{\partial}{\partial\phi}\left(\frac{J^{\frac{1}{2}}}{a\rho}\right)u_\rho u_\phi+\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{a\rho}\right)u_\phi^2 = -\frac{J^{\frac{1}{2}}}{a}\frac{\partial p}{\partial\rho}+\frac{\nu J^{\frac{1}{2}}}{a^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho u_\rho}{J^{\frac{1}{2}}}\right)\right]\right. \\ \left.+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left[J\frac{\partial}{\partial\phi}\left(\frac{u_\rho}{J^{\frac{1}{2}}}\right)\right]+\frac{1}{J^{\frac{1}{2}}}\frac{\partial^2u_\rho}{\partial\zeta^2}\right\}+\frac{\nu\rho}{a^2}\frac{\partial(\rho^{-2}J,u_\phi)}{\partial(\rho,\phi)}, \tag{3.1}$$

$$\frac{du_\phi}{dt}-\rho\frac{\partial}{\partial\rho}\left(\frac{J^{\frac{1}{2}}}{a\rho}\right)u_\rho u_\phi+\frac{\partial}{\partial\phi}\left(\frac{J^{\frac{1}{2}}}{a\rho}\right)u_\rho^2 = \frac{-J^{\frac{1}{2}}}{a\rho}\frac{\partial p}{\partial\phi} \\ +\frac{\nu J^{\frac{1}{2}}}{a^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho u_\phi}{J^{\frac{1}{2}}}\right)\right]+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left[J\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{J^{\frac{1}{2}}}\right)\right]+\frac{1}{J^{\frac{1}{2}}}\frac{\partial^2u_\phi}{\partial\zeta^2}\right\}-\frac{\nu\rho}{a^2}\frac{\partial(\rho^{-2}J,u_\rho)}{\partial(\rho,\phi)}, \tag{3.2}$$

$$\frac{du_\zeta}{dt} = -\frac{1}{a}\frac{\partial p}{\partial \zeta} + \frac{\nu}{a^2}\left\{\frac{J}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u_\zeta}{\partial \rho}\right) + \frac{J}{\rho^2}\frac{\partial^2 u_\zeta}{\partial \phi^2} + \frac{\partial^2 u_\zeta}{\partial \zeta^2}\right\}, \tag{3.3}$$

$$\frac{J}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\rho u_\rho}{J^{\frac{1}{2}}}\right) + \frac{J}{\rho}\frac{\partial}{\partial \phi}\left(\frac{u_\phi}{J^{\frac{1}{2}}}\right) + \frac{\partial u_\zeta}{\partial \zeta} = 0, \tag{3.4}$$

where
$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{J^{\frac{1}{2}}}{a} u_\rho \frac{\partial}{\partial \rho} + \frac{J^{\frac{1}{2}}}{a\rho} u_\phi \frac{\partial}{\partial \phi} + \frac{u_\zeta}{a}\frac{\partial}{\partial \zeta},$$

p denotes the kinematic pressure (pressure/density), u_ρ, u_ϕ and u_ζ the components of velocity, and t the time.

In the present paper it is our object to discuss the linearized theory of instability of the basic flow given by (2.15). To this end we write

$$u_\rho = \tfrac{1}{2}\alpha\epsilon(q_1+q_2)\, U(x,\phi) + (\nu/a\alpha)\, u(x,\phi,\zeta,t), \tag{3.5}$$

$$u_\phi = \tfrac{1}{2}(q_1+q_2)\, V(x,\phi) + (q_1-q_2)\, v(x,\phi,\zeta,t), \tag{3.6}$$

$$u_\zeta = (\nu/a\alpha)\, w(x,\phi,\zeta,t), \tag{3.7}$$

$$p = (\nu q_1/a\alpha^2)\, P(x,\sigma) + (\nu/a^2\alpha^2)\, p'(x,\phi,\zeta,t), \tag{3.8}$$

where $U = [2J^{\frac{1}{2}}/\epsilon\rho(1+\mu)]\,(\partial\Psi/\partial\phi), \quad V = [-2J^{\frac{1}{2}}/(1+\mu)]\,(\partial\Psi/\partial x)$

and P denote the dimensionless basic flow and u, v, w and p' denote the dimensionless perturbation. A few words are perhaps needed in order to explain the scalings assumed above. The basic flow has components in the ρ and ϕ directions, and the coefficient $\alpha\epsilon$ of the ρ component expresses the fact, which follows from (2.8) and details of (2.15), that u_ρ goes to zero with α and with ϵ. The velocity scale $\frac{1}{2}(q_1+q_2)$ is chosen for convenience. Now consider the perturbation quantities. Three-dimensional perturbations of Taylor-vortex type may, in general, depend upon x, ϕ, ζ and t. Moreover, it is known from studies of the nonlinear problem in the concentric case that appropriate velocity scales are $\nu/a\delta$ radially and axially, and q_1-q_2 azimuthally (Stuart 1958; Davey 1962); corresponding scales here are $\nu/a\alpha$ and q_1-q_2 as shown. The scales of the kinematic pressure follow from lubrication theory (DiPrima & Stuart 1972) and from a knowledge of the stability problem in the concentric case.

Linear forms of (3.1)–(3.4) follow if (3.5)–(3.8) are substituted and terms quadratic in u, v, w are ignored. However, before giving those equations we first use the fact that Taylor vortices are periodic along the axis when the cylinders are very long, with a wavelength comparable with the gap between the cylinders (Vohr 1967, 1968). Thus we assume that

$$\left.\begin{aligned} u &= e^{\sigma\tau}u_1(x,\phi)\cos\lambda\xi, & v &= e^{\sigma\tau}v_1(x,\phi)\cos\lambda\xi,\\ w &= e^{\sigma\tau}w_1(x,\phi)\sin\lambda\xi, & p' &= e^{\sigma\tau}p_1(x,\phi)\cos\lambda\xi,\end{aligned}\right\} \tag{3.9}$$

where we have defined

$$\zeta = \alpha\xi, \quad t = (a^2\alpha^2/\nu)\,\tau. \tag{3.10}$$

Clearly σ and λ represent a non-dimensional growth rate and wavenumber, respectively.

The following linearized stability equations then result:

$$\epsilon R_a J^{\frac{1}{2}}\left(U\frac{\partial u_1}{\partial x}+\frac{\partial U}{\partial x}u_1\right)+\frac{R_a J^{\frac{1}{2}}}{\rho}\left(V\frac{\partial u_1}{\partial \phi}+\epsilon c R_a\frac{\partial U}{\partial \phi}v_1\right)$$
$$-R_a\frac{\partial}{\partial \phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)(Vu_1+\epsilon c R_a Uv_1)+\rho\frac{\partial}{\partial \rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)TVv_1$$
$$=-J^{\frac{1}{2}}\frac{\partial p_1}{\partial x}+(JL^2-\lambda^2-\sigma)u_1-\alpha^2 Hu_1+cR_a\rho\left\{\alpha\frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial v_1}{\partial \phi}-\frac{\partial}{\partial \phi}\left(\frac{J}{\rho^2}\right)\frac{\partial v_1}{\partial x}\right\}, \tag{3.11}$$

$$J^{\frac{1}{2}}\left(\frac{1}{c}\frac{\partial V}{\partial x}u_1+\epsilon R_a U\frac{\partial v_1}{\partial x}\right)+\frac{1}{\rho}J^{\frac{1}{2}}R_a\left(V\frac{\partial v_1}{\partial \phi}+\frac{\partial V}{\partial \phi}v_1\right)$$
$$-\alpha\rho\frac{\partial}{\partial \rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)\left(\frac{1}{c}Vu_1+\epsilon R_a Uv_1\right)+2\frac{\alpha^2\epsilon}{c}\frac{\partial}{\partial \phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)u_1$$
$$=-\frac{J^{\frac{1}{2}}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p_1}{\partial \phi}+(JL^2-\lambda^2-\sigma)v_1-\alpha^2 Hv_1$$
$$+\frac{\rho\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial u_1}{\partial \phi}+\frac{\partial}{\partial \phi}\left(\frac{J}{\rho^2}\right)\frac{\partial u_1}{\partial x}\right\}, \tag{3.12}$$

$$\epsilon R_a J^{\frac{1}{2}}U\frac{\partial w_1}{\partial x}+\frac{1}{\rho}J^{\frac{1}{2}}R_a V\frac{\partial w_1}{\partial \phi}=\lambda p_1+(JL^2-\lambda^2-\sigma)w_1, \tag{3.13}$$

$$J^{\frac{1}{2}}\frac{\partial u_1}{\partial x}-\alpha\rho\frac{\partial}{\partial \rho}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)u_1+cR_a\left[\frac{J^{\frac{1}{2}}}{\rho}\frac{\partial v_1}{\partial \phi}-\frac{\partial}{\partial \phi}\left(\frac{J^{\frac{1}{2}}}{\rho}\right)v_1\right]+\lambda w_1=0. \tag{3.14}$$

In these equations we have used the following definitions:

$$L^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\alpha}{1+\alpha x}\frac{\partial}{\partial x}+\left(\frac{\alpha}{1+\alpha x}\right)^2\frac{\partial^2}{\partial \phi^2}, \tag{3.15}$$

$$H\equiv\rho J^{\frac{1}{2}}\left\{\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right\}\left(\frac{J^{\frac{1}{2}}}{\rho}\right), \tag{3.16}$$

$$R_a=\tfrac{1}{2}(1+\mu)R_M=\tfrac{1}{2}(1+\mu)(q_1 a/\nu)\alpha^2, \tag{3.17}$$

$$T=(q_1 a/\nu)^2\alpha^3(1-\mu^2), \tag{3.18}$$

$$c=2[(1-\mu)/(1+\mu)]. \tag{3.19}$$

The boundary conditions on (3.11)–(3.14) are that u_1, v_1 and w_1 are zero at $x=\pm\frac{1}{2}$, and that the solution has period 2π in ϕ. The system of partial differential equations (3.11)–(3.14) with these conditions defines an eigenvalue problem

$$F(\sigma, T, \epsilon, c, \alpha, \lambda)=0. \tag{3.20}$$

The parameter R_a does not appear in (3.20) since R_a can be expressed in terms of T and c. The flow is unstable if there exist solutions of (3.20) with $\operatorname{Re}\sigma>0$. For given values of ϵ, c and α the critical value of T is determined as the minimum value of T for all positive λ such that $\operatorname{Re}\sigma=0$. For the case $\epsilon=0$ all calculations indicate that the condition $\operatorname{Re}\sigma=0$ occurs with $\sigma=0$, and we shall assume that the critical conditions are determined by $\sigma=0$ for $\epsilon\neq 0$. This is consistent with the experimental observations that the instability leads to a steady, rather than oscillatory, secondary flow. It is abundantly clear that (3.11)–(3.14) present a formidable obstacle, at least in the general case. In the next section we seek a means by which they may be simplified, but in such a way that their essential features are retained.

4. An asymptotic approximation to the stability problem

The partial differential equations that we wish to solve, namely (3.11)–(3.14), have coefficients which depend on x and ϕ through U, V, J and ρ. If, however, $\epsilon = 0$, then U, V and J do not depend on ϕ, because the cylinders are then concentric. Then a solution can be sought with $\partial/\partial\phi \equiv 0$ and will describe the usual axisymmetric Taylor-vortex instability and resulting secondary flow. Our object now is to obtain a generalized perturbation which yields a solution distinct from that of the concentric case, at least when ϵ is small; no attempt is made here to discuss (3.11)–(3.14) in their full generality.

For purposes of explanation only of the asymptotic method to be used, the writers have found it helpful to consider the model equation

$$(D^2-\lambda^2)(D^2-\lambda^2-\sigma-R_a V_0(x)[\partial/\partial\phi])^2 v+\lambda^2 T[V_0(x)+\epsilon V_1(x)\cos\phi]v = 0, \quad (4.1)$$

together with appropriate homogeneous boundary conditions, where

$$D \equiv \partial/\partial x \quad (4.2)$$

and $V_0(x)$ and $V_1(x)$ are given. This yields an eigenvalue problem for T as a function of λ, σ, ϵ and R_a. Equation (4.1) contains the essential features of (3.11)–(3.14) in that it models (i) how the basic flow depends on ϕ and (ii) how differentiation with respect to ϕ enters into the problem. Indeed it is clear that, if $\epsilon = 0$ and $\partial/\partial\phi \equiv 0$, then (4.1) has the form of the classical stability equation for the case of a small gap. The essential additions we have made are to allow the basic velocity distribution to depend upon ϕ, and to allow for a ϕ derivative.

Now, if we were to use the 'parallel-flow' assumption, as commonly used in boundary-layer stability, the expression $[V_0(x)+\epsilon V_1(x)\cos\phi]$ would be regarded as a function of x only, with ϕ as a parameter, and $\partial/\partial\phi$ would be disregarded. Then the eigenvalue T of the homogeneous differential system would, implicitly, depend upon ϕ. Calculations of this kind for the case of eccentric cylinders have been made by DiPrima (1963) and Ritchie (1968) and give, essentially, a 'local' criterion for instability, local in the sense of having a value for T for each value of ϕ. For Taylor-vortex instability such analyses really parallel closely Görtler's (1940) treatment of the centrifugal instability of a boundary layer on a concave wall, since he, too, used the parallel-flow (or local) approximation. Such a procedure is not entirely satisfactory, and we would like to obtain for the global problem an eigenvalue T which is independent of ϕ.

The mathematical difficulty is that we must solve the partial differential equations in such a way as to account for the ϕ variation. Since R_a is small in lubrication theory, it appears that the term $R_a V_0(x)\ \partial/\partial\phi$ may be negligible; however, to neglect it brings in the parallel-flow approximation mentioned above and has the implication that T is a function of ϕ. A way out of this difficulty is to allow $\partial/\partial\phi$ to take an appropriate magnitude, relative to the ϕ dependence imposed by the term $\epsilon \cos\phi$. In the actual problem we certainly have α as a small parameter, since this is a concomitant of lubrication theory. Now consider the parameter R_a. Equations (3.17) and (3.18) indicate that

$$R_a \equiv (T\alpha/2c)^{\frac{1}{2}}. \tag{4.3}$$

In the concentric problem T is the stability parameter and c is a constant for given μ. If we keep T fixed, or allow it to vary only in a limited range, then R_a is proportional to $\alpha^{\frac{1}{2}}$. Then, a glance at (4.1) indicates that the response of the term proportional to $\partial/\partial\phi$ is proportional to the ϕ dependence imposed by $\epsilon \cos\phi$ if

$$R_a \sim \alpha^{\frac{1}{2}} \sim \epsilon. \tag{4.4}$$

Thus we set

$$\alpha^{\frac{1}{2}} = k\epsilon(2c)^{\frac{1}{2}}, \tag{4.5}$$

where k is a fixed parameter and $(2c)^{\frac{1}{2}}$ is introduced for convenience. Now, using (4.3) and (4.5) we could develop a solution of (4.1) by expanding $v(x,\phi)$, T and λ in powers of ϵ. This would give a solution for both ϵ and α small, but subject to the over-riding relation (4.5). Instead of pursuing the model equation (4.1), which is of no physical interest, we now return to the true stability equations for our problem, but still make use of the mathematical idea explained above.

We now use (4.3) and (4.5) in (3.11)–(3.14) and then expand the perturbation velocity and pressure field as follows:

$$u_1 = u_{10}(x,\phi) + \epsilon u_{11}(x,\phi) + \epsilon^2 u_{12}(x,\phi) + \dots, \tag{4.6}$$

$$v_1 = v_{10}(x,\phi) + \epsilon v_{11}(x,\phi) + \epsilon^2 v_{12}(x,\phi) + \dots, \tag{4.7}$$

$$w_1 = w_{10}(x,\phi) + \epsilon w_{11}(x,\phi) + \epsilon^2 w_{12}(x,\phi) + \dots, \tag{4.8}$$

$$p_1 = p_{10}(x,\phi) + \epsilon p_{11}(x,\phi) + \epsilon^2 p_{12}(x,\phi) + \dots. \tag{4.9}$$

Moreover, we let

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \dots. \tag{4.10}$$

For the calculation of the critical value of T at which instability may occur, it can be shown that the variation of λ from its 'critical' value at $\epsilon = 0$ does not affect T through terms of the order of ϵ^2 (a related argument for another problem is given by Chandrasekhar 1961, p. 313). Since we take our calculations of T only to order ϵ^2 we shall keep λ fixed. The functions in (4.6)–(4.9) depend in various ways upon σ, λ, k, c, T_0, T_1, $T_2 \dots$.

In association with these expansions we must recognize that U and V of (3.5) and (3.6) depend upon α, R_a and ϵ, so with use of (3.17), (4.3) and (4.5), they too can be expanded in powers of ϵ. In particular we need $R_M = 2T^{\frac{1}{2}}k\epsilon/(1+\mu)$; we then have

$$V = V_0(x) + \epsilon V_1(x,\phi) + \epsilon^2[V_{20}(x) + kT^{\frac{1}{2}}V_{21}(x,\phi) + k^2c^2V_{22}(x)] + O(\epsilon^3), \tag{4.11}$$

$$U = U_0(x,\phi) + O(\epsilon). \tag{4.12}$$

In these formulae V_0, V_1, V_{20} and U_0 come from Ψ_{00}, while V_{21} comes from Ψ_{10} and V_{22} from Ψ_{01} of (2.15). As noted at the end of § 2, the term $R_M^2\,\Psi_{20}$ in (2.15) is of $O(\alpha\epsilon)$, which is now seen to be equivalent to $O(\epsilon^3)$ and therefore negligible to the order of (4.11). Moreover, it can be shown from DiPrima & Stuart (1972) that

$$V_0 = 1 - cx, \tag{4.13}$$

$$V_1 = 6(x^2 - \tfrac{1}{4}) \cos\phi, \tag{4.14}$$

$$V_{20} = 3(x^2 - \tfrac{1}{4}), \tag{4.15}$$

$$V_{21} = (x^2 - \tfrac{1}{4}) [\tfrac{1}{2}(1 - \tfrac{1}{20}c^2)(\tfrac{1}{20} - x^2) - \tfrac{1}{5}cx(\tfrac{7}{12} - x^2)] \sin\phi, \tag{4.16}$$

$$V_{22} = x^2 - \tfrac{1}{4}, \tag{4.17}$$

$$U_0 = 2(x^2 - \tfrac{1}{4})(x - \tfrac{1}{4}c) \sin\phi. \tag{4.18}$$

It may be noted that the approximation (4.11) preserves the property of basic flow separation provided that $\epsilon \geqslant 0{\cdot}28$, a result which should be compared with (2.17). These formulae are sufficient to take the stability calculations up to order ϵ^2.

Using (4.2)–(4.18) we obtain sets of simplified differential equations for the functions arising in (4.6)–(4.9), and these must be solved subject to no-slip conditions and the requirement that the solution be single-valued in ϕ. The solutions are given in § 5.

5. Solution of the simplified equations

Order ϵ^0

By elimination of p_{10} and w_{10} we have

$$\lambda^2 T_0 V_0 v_{10} = NMu_{10}, \quad -u_{10} = Nv_{10}, \tag{5.1}$$

where $$M \equiv (D^2 - \lambda^2), \quad N \equiv (D^2 - \lambda^2 - \sigma),$$

with the boundary conditions

$$u_{10} = v_{10} = Du_{10} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.2}$$

This differential system yields an eigenrelation between λ, T_0 and σ. For example, for $c = 2$ ($q_2 = 0$, outer cylinder at rest), the critical Taylor number (T) is about $1694{\cdot}95$ at $\lambda = 3{\cdot}13$, and occurs (it is believed) with σ real and therefore equal to zero (Davey 1962).

We write the solution of (5.1) and (5.2) in the form

$$u_{10} = -B(\phi) f_0(x), \quad v_{10} = B(\phi) g_0(x), \tag{5.3}$$

where f_0 and g_0 together provide the eigenfunction of the system of *ordinary* differential equations

$$M^2 f_0 + \lambda^2 T_0 V_0 g_0 - \sigma M f_0 = 0, \tag{5.4}$$

$$f_0 - Mg_0 + \sigma g_0 = 0, \tag{5.5}$$

$$f_0 = Df_0 = g_0 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.6}$$

However, since ϕ is a parameter as far as (5.4)–(5.6) are concerned, we can and must allow the arbitrary multiplicative constant to be a function of ϕ. The determination of $B(\phi)$ requires the consideration of higher order terms in the expansion.

Adjoint equation: order ϵ^0

Later we shall need to use the differential system adjoint to the system (5.4)–(5.6). This is given by

$$M^2 f_0^+ + g_0^+ - \sigma M f_0^+ = 0, \tag{5.7}$$

$$\lambda^2 T_0 V_0 f_0^+ - M g_0^+ + \sigma g_0^+ = 0, \tag{5.8}$$

$$f_0^+ = Df_0^+ = g_0^+ = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.9}$$

The eigenvalues T_0, λ, σ and c are identical to those of (5.4)–(5.6) but the adjoint eigenfunction pair (f_0^+, g_0^+) is different from (f_0, g_0), as the form of the equations indicates.

Order ϵ

By elimination of p_{11} and w_{11}, use of the properties of the system to $O(\epsilon^0)$, and with the definition

$$\bar{u}_{11} = -u_{11}, \tag{5.10}$$

we obtain

$$M^2\bar{u}_{11} + \lambda^2 T_0 V_0 v_{11} - \sigma M \bar{u}_{11} - \; = B(\phi)\cos\phi\, F_{111}(x) + kT_0^{\frac{1}{2}}\frac{dB}{d\phi}F_{112}(x) + T_1 B(\phi) F_{113}(x), \tag{5.11}$$

$$\bar{u}_{11} - Mv_{11} + \sigma v_{11} = B(\phi)\cos\phi\, G_{111}(x) + kT_0^{\frac{1}{2}}\frac{dB}{d\phi}G_{112}(x), \tag{5.12}$$

$$\bar{u}_{11} = D\bar{u}_{11} = v_{11} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.13}$$

Here

$$F_{111}(x) = -6\lambda^2 T_0(x^2 - \tfrac{1}{4})\, g_0 + 4(M - \tfrac{1}{2}\sigma) D^2 f_0, \tag{5.14}$$

$$F_{112}(x) = V_0 M f_0, \quad F_{113}(x) = -\lambda^2 V_0 g_0, \qquad (5.15), (5.16)$$

$$G_{111}(x) = [(12/c)x + 1] f_0 - 2D^2 g_0, \quad G_{112}(x) = -V_0 g_0. \qquad (5.17), (5.18)$$

Once the system to $O(\epsilon^0)$ has been solved, the right-hand sides of (5.11) and (5.12) are known, except for the value of T_1, as yet unknown, and for the function $B(\phi)$, as yet neither assigned nor determined. The two operators on the left-hand sides of (5.11) and (5.12) have the same forms as those in (5.4) and (5.5). Moreover, differentials with respect to ϕ do not occur in those operators; consequently, ϕ may be regarded as a parameter as far as (5.11) and (5.12) are concerned.

It is at this point that the adjoint function pair (f_0^+, g_0^+) comes into its own. It is well known that, since the homogeneous system (5.11)–(5.13) has a non-trivial solution, the non-homogeneous problem will have a solution only if an appropriate orthogonality condition involving the solution of the adjoint problem is satisfied. In this problem the orthogonality condition requires that when we multiply the right-hand sides of (5.11) and (5.12) by f_0^+ and g_0^+ respectively, add and integrate over $(-\frac{1}{2}, \frac{1}{2})$ the result must be zero. This gives a relation between $B(\phi)$, $dB/d\phi$, $\cos\phi$, k and T_1, namely

$$kT_0^{\frac{1}{2}}\frac{dB}{d\phi}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{112} + g_0^+ G_{112})\,dx + \cos\phi\, B(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{111} + g_0^+ G_{111})\,dx + T_1 B(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{113}\,dx = 0. \tag{5.19}$$

This of course, is a first-order linear differential equation for $B(\phi)$. The coefficients can be evaluated in terms of the $O(\epsilon^0)$ solution and of the adjoint solution.

Upon solving (5.19) we find immediately that, if $B(\phi)$ is to be single-valued, then

$$T_1 \equiv 0. \tag{5.20}$$

The solution for $B(\phi)$ then follows:

$$B(\phi) = B_0(k) \exp[\Gamma/kT_0^{\frac{1}{2}}) (\sin\phi - 1)], \tag{5.21}$$

where

$$\Gamma = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ f_0^+[-6\lambda^2 T_0(x^2 - \frac{1}{4}) g_0 + 4(M - \frac{1}{2}\sigma) D^2 f_0] + g_0^+ \left[\left(\frac{12x}{c} + 1\right) f_0 - 2D^2 g_0\right]\right\} dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_0^+ M f_0 - g_0^+ g_0] V_0\, dx} \tag{5.22}$$

and $B_0(k)$ is a constant which cannot be determined within the framework of a linear theory and will, of course, depend upon all the parameters of the problem; the dependence on k is shown explicitly.

With use of (5.20) and (5.21) it then follows that (5.11) and (5.12) can be solved in the form

$$-u_{11} = \bar{u}_{11} = B(\phi)\cos\phi f_1(x) + B_1(\phi) f_0(x), \tag{5.23}$$

$$v_{11} = B(\phi)\cos\phi\, g_1(x) + B_1(\phi)\, g_0(x), \tag{5.24}$$

where f_1 and g_1 satisfy the system of ordinary differential equations

$$M^2 f_1 + \lambda^2 T_0 g_1 - \sigma M f_1 = -6\lambda^2 T_0(x^2 - \tfrac{1}{4}) g_0 + 4(M - \tfrac{1}{2}\sigma) D^2 f_0 + \Gamma V_0 M f_0, \tag{5.25}$$

$$f_1 - M g_1 + \sigma g_1 = \left(\frac{12x}{c} + 1\right) f_0 - 2D^2 g_0 - \Gamma V_0 g_0, \tag{5.26}$$

$$f_1 = Df_1 = g_1 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.27}$$

Formula (5.22) ensures that this differential system has a solution. Of course the solution of (5.25)–(5.27) is determined only up to an additive multiple of the eigenfunction of the corresponding homogeneous system, and this is reflected by the inclusion of the terms $B_1(\phi) f_0(x)$ and $B_1(\phi)\, g_0(x)$ in (5.23) and (5.24), respectively. To determine the function $B_1(\phi)$ requires consideration of the next higher term in the expansion.

Order ϵ^2

By elimination of p_{12} and w_{12} and use of certain properties of the systems to order ϵ^0 and ϵ, together with the definition

$$u_{12} = -\bar{u}_{12}, \tag{5.28}$$

we obtain

$$\begin{aligned} M^2\bar{u}_{12} + \lambda^2 T_0 V_0 v_{12} - \sigma M \bar{u}_{12} = {} & B_1(\phi)\cos\phi F_{111}(x) + kT_0^{\frac{1}{2}} \frac{dB_1}{d\phi} F_{112}(x) + B(\phi)\, F_{123}(x) \\ & + k^2 B(\phi) F_{124}(x) + B(\phi)\cos 2\phi\, F_{125}(x) + kT_0^{\frac{1}{2}} B(\phi)\sin\phi F_{126}(x) \\ & + T_2 B(\phi)\, F_{127}(x), \end{aligned} \tag{5.29}$$

$$\bar{u}_{12} - Mv_{12} + \sigma v_{12} = B_1(\phi)\cos\phi\, G_{111}(x) + kT_0^{\frac{1}{2}}\frac{dB_1}{d\phi}G_{112}(x)$$
$$+ B(\phi)\,G_{123}(x) + k^2B(\phi)\,G_{124}(x) + B(\phi)\cos 2\phi\, G_{125}(x) + kT_0^{\frac{1}{2}}B(\phi)\sin\phi\, G_{126}(x). \tag{5.30}$$

The boundary conditions are

$$\bar{u}_{12} = D\bar{u}_{12} = v_{12} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{5.30a}$$

In these equations certain functions are given by (5.13)–(5.17), while the others are as follows:

$$F_{123}(x) = \tfrac{1}{2}\Gamma[6(x^2-\tfrac{1}{4})\,Mf_0 - V_0(3D^2-\lambda^2)f_0 - 12f_0 + 2cDf_0]$$
$$-(3D^2-\lambda^2-\tfrac{1}{2}\sigma)\,D^2f_0 - 3\lambda^2T_0(x^2-\tfrac{1}{4})\,g_0 - \Gamma cNDg_0$$
$$+2(M-\tfrac{1}{2}\sigma)\,D^2f_1 - 3\lambda^2T_0(x^2-\tfrac{1}{4})\,g_1 + \tfrac{1}{2}\Gamma V_0 Mf_1, \tag{5.31}$$

$$F_{124}(x) = -4c(M-\tfrac{1}{2}\sigma)\,Df_0 + 2\lambda^2T_0c(x+\tfrac{1}{2})\,V_0g_0 - \lambda^2T_0c^2V_{22}g_0, \tag{5.32}$$

$$F_{125}(x) = \tfrac{1}{2}\Gamma[6(x^2-\tfrac{1}{4})\,Mf_0 - V_0(3D^2-\lambda^2)f_0 - 12f_0 + 2cDf_0]$$
$$-(3D^2-\lambda^2-\tfrac{1}{2}\sigma)\,D^2f_0 - \Gamma cNDg_0 + 2(M-\tfrac{1}{2}\sigma)\,D^2f_1$$
$$+\tfrac{1}{2}\Gamma V_0Mf_1 - 3\lambda^2T_0(x^2-\tfrac{1}{4})\,g_1, \tag{5.33}$$

$$F_{126}(x) = 2(x^2-\tfrac{1}{4})\,(x-\tfrac{1}{4}c)\,MDf_0 + (6x^2-cx-\tfrac{1}{2})\,Mf_0$$
$$+V_0(D^2+\lambda^2)\,f_0 + 12f_0 + 12xDf_0$$
$$-\lambda^2T_0(x^2-\tfrac{1}{4})\,[\tfrac{1}{2}(1-\tfrac{1}{12}c^2)\,(\tfrac{1}{20}-x^2) - \tfrac{1}{5}cx(\tfrac{7}{12}-x^2)]\,g_0$$
$$-3c(M-\tfrac{1}{3}\sigma)\,Dg_0 - V_0Mf_1, \tag{5.34}$$

$$F_{127}(x) = -\lambda^2V_0g_0, \tag{5.35}$$

$$G_{123}(x) = -\tfrac{1}{2}f_0 + \tfrac{3}{2}D^2g_0 + \tfrac{1}{2}\Gamma[1-cx-6(x^2-\tfrac{1}{4})]\,g_0 + \tfrac{1}{2}f_1 + (6x/c)f_1 - D^2g_1 - \tfrac{1}{2}\Gamma V_0g_1, \tag{5.36}$$

$$G_{124} = 2f_0 + 2cDg_0, \tag{5.37}$$

$$G_{125}(x) = -(6x/c)f_0 + \tfrac{1}{2}D^2g_0 + \tfrac{1}{2}\Gamma[1-cx-6(x^2-\tfrac{1}{4})]\,g_0 + \tfrac{1}{2}f_1$$
$$+(6x/x)\,f_1 - D^2g_1 - \tfrac{1}{2}\Gamma V_0g_1, \tag{5.38}$$

$$G_{126}(x) = (1/c)f_0[\tfrac{1}{2}(1-\tfrac{1}{12}c^2)\,(-4x^3+\tfrac{3}{5}x)] + f_0(x^4-\tfrac{1}{2}x^2+\tfrac{7}{240})$$
$$-2(x^2-\tfrac{1}{4})\,(x-\tfrac{1}{4}c)\,Dg_0 + 6g_0(x^2-\tfrac{1}{4}) + V_0g_1. \tag{5.39}$$

Again, in order for the system (5.29)–(5.30a) to have a solution we require that, when we multiply the right-hand sides of (5.29) and (5.30) by f_0^+ and g_0^+, respectively, add and integrate over $(-\frac{1}{2}, \frac{1}{2})$, the result must be zero. Thus we obtain the following first-order linear differential equation for $B_1(\phi)$:

$$\left[kT_0^{\frac{1}{2}}\frac{dB_1(\phi)}{d\phi} - \Gamma B_1(\phi)\cos\phi\right]\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{112} + g_0^+G_{112})\,dx$$
$$+B(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{123} + g_0^+G_{123})\,dx$$
$$+k^2B(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{124} + g_0^+G_{124})\,dx + B(\phi)\cos 2\phi\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{125} + g_0^+G_{125})\,dx$$
$$+kT_0^{\frac{1}{2}}B(\phi)\sin\phi\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{126} + g_0^+G_{126})\,dx + T_2B(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}f_0^+F_{127}\,dx = 0. \tag{5.40}$$

The integral coefficients are known from properties of the functions of order ϵ^0 and ϵ.

The constant T_2 is determined by the requirement that the solution of (5.40) be single-valued in ϕ. This gives

$$T_2 = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{123} + g_0^+ G_{123})\,dx + k^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{124} + g_0^+ G_{124})\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{127}\,dx}. \tag{5.41}$$

The function $B_1(\phi)$ is then given by

$$B_1(\phi) = B(\phi)\,[\Gamma_1 \cos\phi + (\Gamma_2/2kT_0^{\frac{1}{2}}) \sin 2\phi], \tag{5.42}$$

where

$$\Gamma_1 = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{126} + g_0^+ G_{126})\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{112} + g_0^+ G_{112})\,dx}, \tag{5.43}$$

$$\Gamma_2 = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{125} + g_0^+ G_{125})\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{112} + g_0^+ G_{112})\,dx}. \tag{5.44}$$

An integration constant inside the square brackets of (5.42) has been set equal to zero, since to retain it would merely involve a redefinition of the amplitude B_0 of (5.21), as (5.3), (5.23) and (5.24) indicate. In solving for (f_1, g_1) a convention is needed as to the 'amount' of (f_0, g_0) which is included in (f_1, g_1). However, a calculation shows that v_{11} and u_{11} are independent of the convention selected, as is the value of T_2.

We are now able to give the form of the velocity field including terms up to order ϵ, and the critical Taylor number to order ϵ^2. To order ϵ the velocity field is given by

$$u_1 = -B(\phi)\,[f_0(x) + \epsilon f_1(x)\cos\phi] - \epsilon B_1(\phi) f_0(x), \tag{5.45}$$

$$v_1 = B(\phi)\,[g_0(x) + \epsilon g_1(x)\cos\phi] + \epsilon B_1(\phi)\, g_0(x), \tag{5.46}$$

$$w_1 = \lambda^{-1} B(\phi)\, Df_0 + \epsilon\lambda^{-1}[B(\phi)\cos\phi Df_1 + (B_1(\phi) - B(\phi)\cos\phi)\, Df_0 - c\Gamma\, B(\phi)\cos\phi\, g_0]. \tag{5.47}$$

Using (5.24) we may re-write these formulae as

$$u_1 = -B(\phi)\{f_0(x) + \epsilon[f_1 \cos\phi + f_0(\Gamma_1\cos\phi + (\Gamma_2/2kT_0^{\frac{1}{2}})\sin 2\phi)]\}, \tag{5.48}$$

$$v_1 = B(\phi)\{g_0(x) + \epsilon[g_1\cos\phi + g_0(\Gamma_1\cos\phi + (\Gamma_2/2kT_0^{\frac{1}{2}})\sin 2\phi)]\}, \tag{5.49}$$

$$w_1 = \lambda^{-1} B(\phi)\{Df_0(x) + \epsilon[Df_1\cos\phi + Df_0(\Gamma_1 - 1)\cos\phi + Df_0\,(\Gamma_2/2kT_0^{\frac{1}{2}})\sin 2\phi - c\Gamma g_0\cos\phi]\}. \tag{5.50}$$

The Taylor number is given by

$$T = T_0 + \epsilon^2(T_{21} + k^2 T_{22}) \tag{5.51}$$

to order ϵ^2, where $(T_{21} + k^2T_{22})$ represents (5.41) and λ, k, c and σ are fixed. Numerical details of the above formulae are given in the next section and compared with experiment.

With knowledge of $B_1(\phi)$ we are in a position to solve (5.29) and (5.30) for $\bar{u}_{12}$ and v_{12}, but the solution can only be determined up to an additive multiple of the eigenfunction (f_0, g_0) in the form $(B_2(\phi) f_0(x), B_2(\phi) g_0(x))$. In order to determine $B_2(\phi)$ it is necessary to go to terms of $O(\epsilon^3)$. Thus we cannot yet calculate the form of the $O(\epsilon^2)$ term in the velocity field, since we have not discussed the determination of $B_2(\phi)$.

6. Detailed results and comparison with experiment

Detailed computations, based on the analysis of the previous section, have generously been performed for the authors by Dr P. M. Eagles of the City University, London. The results are all for the case when the outer cylinder is at rest ($q_2 = 0$), and for neutral stability with $\sigma = 0$. For $\sigma = 0$, $c = 2$ and $\epsilon = 0$ the critical values of T and λ are taken to be

$$T_0 = 1695, \quad \lambda = 3{\cdot}127. \tag{6.1}$$

The calculations were performed by a Runge–Kutta integration routine, using 20 steps and, as a check 30 or 40 steps. Where possible comparison was made with previous work (Davey 1962; Davey, DiPrima & Stuart 1968; Eagles 1971); in other cases independent checks were made. Accuracy is to four significant figures. The main results follow from the solution of (5.4)–(5.6) for the basic eigenfunction pair (f_0, g_0) with λ and T_0 as in (6.1), from (5.7)–(5.9) for the adjoint function pair (f_0^+, g_0^+), from (5.22) for the fundamental constant Γ, from (5.25)–(5.27) for the function pair (f_1, g_1), and from (5.41)–(5.44) for the important constants T_{21}, T_{22}, Γ_1 and Γ_2. The values of several of these quantities are as follows:

$$\Gamma = 23{\cdot}09, \quad T_{21} = -635{\cdot}7, \quad T_{22} = 7877, \qquad (6.2), (6.3), (6.4)$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_0^+ F_{112} + g_0^+ G_{112}]\, dx = (-3{\cdot}799)\, 10^{-5}, \tag{6.5}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_0^+ F_{125} + g_0^+ G_{125}]\, dx = (6{\cdot}898)\, 10^{-4}, \tag{6.6}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} [f_0^+ F_{126} + g_0^+ G_{126}]\, dx = (-9{\cdot}610)\, 10^{-5}, \tag{6.7}$$

where functions are normalized so that $Dg_0^+ = D^3 f_0 = 1$, $x = -\frac{1}{2}$, which implies that $g_0 = (1{\cdot}71814)\, 10^{-4}$, $x = 0$ and $D^2 f_0^+ = (-1{\cdot}30764)\, 10^{-2}$, $x = -\frac{1}{2}$. Thus, from (5.43), (5.44) and (6.5)–(6.7) we have

$$\Gamma_1 = 2{\cdot}530, \quad \Gamma_2 = 18{\cdot}16. \qquad (6.8), (6.9)$$

A matter of some importance is that the parameter k, defined by (4.5), and which appears in a rather simple way in the formulae (5.48)–(5.51), is left free. The implications of varying this parameter will be discussed later.

Taylor number

We note that (5.51), together with (6.3) and (6.4), yields

$$T = 1695 + \epsilon^2(-635\cdot7 + 7877k^2). \tag{6.10}$$

We emphasize that T is given by

$$T = (q_1 a/\nu)^2 \alpha^3 \tag{6.11}$$

for $\mu = 0$ $(c = 2)$, where α is related to δ and ϵ by (2.14). However, this Taylor number contains an implicit dependence on ϵ through α. We eliminate this by defining the more conventional Taylor number

$$T_a = (q_1 a/\nu)^2 \delta^3. \tag{6.12}$$

Using (2.14), we obtain

$$T = T_a(1-\epsilon^2)^{\frac{3}{2}}\{1 - \tfrac{1}{2}\delta[1-(1-\epsilon^2)^{\frac{1}{2}}]\}^3. \tag{6.13}$$

Substituting this expression for T in (6.10), noting that $k^2\epsilon^2 = \frac{1}{4}\alpha$ from (4.5), and expanding for ϵ and δ small and neglecting higher order terms, we obtain the following from (6.10):

$$T_a = 1695(1 + 1\cdot162\delta) + 1907\epsilon^2 = 1695(1 + 1\cdot162\delta)(1 + 1\cdot125\epsilon^2) + O(\delta\epsilon^2, \delta^2, \epsilon^4). \tag{6.14}$$

Since k has been eliminated from (6.14) we can regard it as a formula which gives the dependence of T_a on the two small parameters δ and ϵ^2, which may now vary independently. The term of order δ comes from the term of $O(\epsilon^2 k^2)$ in (6.10); in fact it has nothing to do with eccentricity, but represents the change of the critical value of T_a in (6.12), due to the small curvature effect. A comparison for $\epsilon = 0$ by interpolation from the work of Roberts (1965), who calculated stability properties including critical Taylor numbers for a variety of values of δ, shows that the coefficient 1·162 of (6.14) is about 1·167 in Roberts' calculations. (A similar comparison with Taylor's (1923) paper gives 1·14, indicating the great accuracy of that classical work of 50 years ago.)

The true effect of eccentricity in (6.14) is represented by the term $1\cdot125\epsilon^2$; this arises from two sources, one being the negative term in (6.10) and the other the effect of the factor $(1-\epsilon^2)^{\frac{3}{2}}$ in (6.13). As can be seen, the latter is much the more important contribution, since it changes the sign of the explicit ϵ^2 term in going from (6.10) to (6.14).

We are now in a position to compare our calculations with the experimental results, particularly with those of Vohr (1968) for $\delta = 0\cdot0104$, because this is the smallest value of δ available; however reference will be made also to results for larger values of δ. Vohr plots the experimental values of the square root of the Taylor number, $T_v = T_a(1+\frac{1}{2}\delta)^{-1}$, which we give as

$$T_v^{\frac{1}{2}} = 41\cdot17(1 + 0\cdot331\delta)(1 + 0\cdot5625\epsilon^2) + O(\delta\epsilon^2, \delta^2, \epsilon^4), \tag{6.15}$$

and shows that T_v rises with ϵ, a feature shown also by experiments of Cole (1965) and Kamal (1966). Figure 2 shows a comparison of (6.15) with the measurements for $\delta = 0\cdot0104$. It can be seen that the agreement is excellent for values of

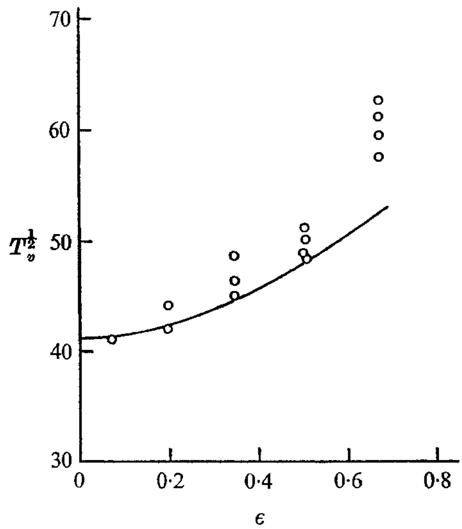

FIGURE 2. Comparison of theory with experimental measurements of Vohr (1968) for $\delta = 0{\cdot}0104$. ———, theory; ○, experiment.

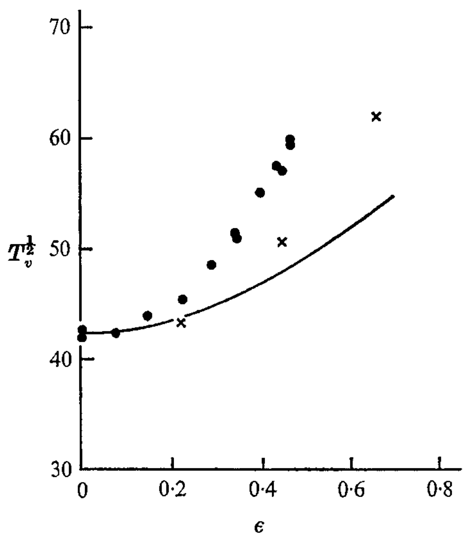

FIGURE 3. Comparison of theory with experimental measurements. ———, theory. Experiments: ●, Vohr (1968) for $\delta = 0{\cdot}099$ using torque measurements; ×, Kamal (1966) for $\delta = 0{\cdot}0904$ using visual observation with aluminium powder.

ϵ up to about 0·5. Presumably the term of order ϵ^4 is needed in (6.10), (6.14) and (6.15) in order to account for the experimental value of the critical Taylor number at values of ϵ larger than 0·5.

For $\delta = 0{\cdot}099$ our result is shown in figure 3, together with the measurements of Vohr. Agreement in this case is good for values of ϵ up to about 0·2 only, indicating perhaps that the $O(\epsilon)^4$ effect in (6.15) is much greater in this case. Also shown in figure 3 are the observations of Kamal for $\delta = 0{\cdot}0904$. (The effects of

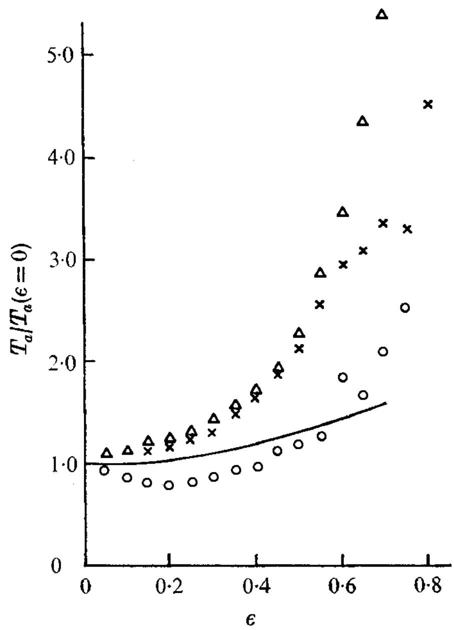

FIGURE 4. Comparison of theory with experimental measurements of Castle & Mobbs (1968) for $\delta = 0{\cdot}112$ (torque) and $\delta = 0{\cdot}0962$ (dye or aluminium). ———, theory. Experiment: ○, first instability (dye); △, second instability (aluminium); ×, second instability (torque).

the small difference in δ, and of the fact that Kamal's Taylor number is $T_a^{\frac{1}{2}}$, are not large.) As can be seen these observations agree much better with our theory. However, it should be noted that Kamal used visual observation while Vohr used torque measurements in determining the critical speed. Dr Vohr has pointed out to us that for his experimental work he believed that torque measurements gave a more accurate measurement of the critical speed than visual observations and that the latter tended to give lower values.

Castle & Mobbs (1968) made measurements for $\delta = 0{\cdot}112$ (torque apparatus) and for $\delta = 0{\cdot}0962$ (visualization by dye or aluminium particles), and their results are shown in figure 4. The solid curve represents $T_a/T_a(\epsilon = 0) = 1 + 1{\cdot}125\epsilon^2$, derivable from (6.14); it is independent of δ in this approximation. The results of Castle & Mobbs obtained from torque measurements and from use of aluminium particles agree with Vohr's, but they observed also a lower incipient mode of instability by means of dye. This dip below T_0, which has been observed also by Versteegen & Jankowski (1969) and Frêne & Godet (1971), agrees with DiPrima's (1963) local theory, and it seems possible therefore that this 'incipient' mode may be a manifestation of local instabilities. Alternatively it may represent the occurrence of another mode which cannot be explained by the present global theory.

The position of maximum vortex activity

Vohr's visual observations of the flow between eccentric cylinders were made with the larger value of δ, namely $\delta = 0{\cdot}099$. They are stated in terms of the angle, denoted here by Θ, measured around the outer cylinder (in the direction of rotation) from the place of maximum gap. In order to compare his observations with our theory we need first to relate Θ to the angle ϕ used in the present work, and originally introduced by the conformal transformation (2.13). Actually there are three angles involved, namely ϕ of the bi-polar co-ordinate system, the angle θ for polar co-ordinates centred on the inner cylinder and the angle Θ for polar co-ordinates centred on the outer cylinder. The relationships on the outer cylinders can be established as follows.

The outer cylinder (see figure 1) is defined by

$$r^2 - 2rae\cos\theta + a^3e^2 - b^2 = 0. \tag{6.16}$$

Also, on the outer cylinder ($\rho = \beta$) the length element is given by

$$ds = bd\Theta = a\left[1 + \frac{\alpha}{(1-\epsilon^2)^{\frac{1}{2}}}(1+\epsilon\cos\theta)\right]d\theta = \frac{a\beta}{J^{\frac{1}{2}}}\bigg|_{x=\frac{1}{2}} d\phi. \tag{6.17}$$

The latter part of this statement follows from (6.16) with J given by (2.33)–(2.35). Using the Sommerfeld transformation (of lubrication theory for a journal bearing), namely

$$\cos\chi = \frac{\cos\phi - \epsilon}{1-\epsilon\cos\phi}, \tag{6.18}$$

we can integrate (6.17) to obtain

$$s = b\Theta = a\left[\theta + \frac{\alpha}{(1-\epsilon^2)^{\frac{1}{2}}}(\theta + \epsilon\sin\theta)\right] = \chi + \frac{\alpha}{(1-\epsilon^2)^{\frac{1}{2}}}(\chi + \tfrac{1}{2}\epsilon\sin\chi), \tag{6.19}$$

to order α, where s and Θ are measured from $\phi = \theta = \chi = 0$. Solving for θ and Θ and using (2.14), we obtain

$$\theta = \chi - \tfrac{1}{2}\epsilon\delta\sin\chi + O(\delta^2), \tag{6.20}$$

$$\Theta = \chi + \tfrac{1}{2}\delta\epsilon\sin\chi + O(\delta^2). \tag{6.21}$$

Comparative values of ϕ, θ, χ and Θ, approximated for ϵ and δ small, are given in table 1. A further useful fact, which is not difficult to check, is that $d\beta/d\Theta$ and $d\beta/d\phi$ are zero together.

Now, if terms of order ϵ are ignored, formulae (5.48)–(5.50) show that the *linearized* theory solution is proportional to $B(\phi)$. From (5.21) this clearly has its maximum at $\phi = \frac{1}{2}\pi$, which, according to table 1 with ϵ terms ignored, corresponds to $\Theta = \frac{1}{2}\pi$. This result, which is valid when δ and ϵ tend to zero and are linked by (2.14) and (4.5), is very significant since it shows the global, or non-local, property of this stability problem. The maximum vortex activity lies not at $\Theta = 0$, where the instability will first occur according to a local theory, but rather at a position displaced by $\frac{1}{2}\pi$ around the annulus, in the direction of rotation.

ϕ	χ	θ	Θ
0	0	0	0
$\frac{1}{2}\pi$	$\frac{1}{2}\pi+\epsilon$	$\frac{1}{2}\pi+\epsilon-\frac{1}{2}\delta\epsilon$	$\frac{1}{2}\pi+\epsilon+\frac{1}{2}\delta\epsilon$
π	π	π	π

TABLE 1

In Vohr's (1968) experiment with $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$, he observed that "the vortices were...most strongly developed...50°[Θ] downstream of maximum clearance". It is of interest to attempt to calculate an $O(\epsilon)$ correction to our asymptotic result $\Theta = \frac{1}{2}\pi$, to see if the position of maximum activity is displaced towards smaller angles. To this end we need to use the rather complicated formulae (5.48)–(5.50).

It is not clear just what property of the flow field contributes most to Vohr's statement. However, one measure of the vortex activity which is easily calculated and which may correspond with observation is the axial velocity in the neighbourhood of the outer cylinder. Since $w_1 = 0$ on the outer cylinder it is clear from a Taylor series expansion that near the outer wall the axial velocity is proportional to the radial gradient $\partial w_1/\partial x$ at $x = \frac{1}{2}$. A calculation shows that this quantity has its maximum at

$$\phi = \tfrac{1}{2}\pi - \phi_2,$$

where

$$\phi_2 = \epsilon\left[\frac{kT_0^{\frac{1}{2}}}{\Gamma}\frac{D^2f_1}{D^2f_0}+\frac{kT_0^{\frac{1}{2}}}{\Gamma}(\Gamma_1-1)+\frac{\Gamma_2}{\Gamma}-kcT_0^{\frac{1}{2}}\frac{Dg_0}{D^2f_0}\right]_{x=\frac{1}{2}}. \tag{6.22}$$

Dr Eagles's computations show that at the outer cylinder

$$D^2f_0 = -(1{\cdot}00)\,10^{-1},\quad D^2f_1 = (1{\cdot}00)10^{-1},\quad Dg_0 = (-4{\cdot}47)10^{-4}. \tag{6.23}$$

Thus for $c = 2$ we obtain

$$\phi_2 = \epsilon(0{\cdot}577k+0{\cdot}786), \tag{6.24}$$

while the corresponding value of Θ is

$$\Theta = \tfrac{1}{2}\pi+\epsilon(0{\cdot}214\text{–}0{\cdot}577k)+O(\delta\epsilon). \tag{6.25}$$

This is less than $\frac{1}{2}\pi$ if $k > 0{\cdot}37$.

The value of k is determined by the choice of δ and ϵ from (4.5). For Vohr's experiments with $\delta = 0{\cdot}099$ and $\epsilon = 0{\cdot}475$ we find that $k = 0{\cdot}33$. For this value of k, formula (6.25) indicates a displacement from 90° away from the observed 50°. In making our comparison with Vohr's observations of the region of maximum vortex strength, we must note that his δ and ϵ, but especially ϵ, may be too large for the asymptotic theory to be completely correct. Moreover, we note that the observation of about 50° was certainly qualitative and 'subjective', as Dr Vohr has explained to us. Thus more work, both experimental and theoretical, is required in order to establish the effects of the magnitudes of δ and ϵ on the position of maximum vortex activity. It is possible also that finite amplitude effects may be relevant.

One further matter ist hat for $\epsilon = 0{\cdot}475$ the basic flow certainly has a zone of separation. Dr Vohr has also indicated that the axial line of reattachment of the basic flow was rather close to the 50° line, and may have affected the observations. From the theoretical point of view we are encouraged about the accuracy of basic flow given by (4.11)–(4.18) by the fact that formula (4.11) for V gives the possibility of separation of the basic flow only if $\epsilon > 0{\cdot}28$, a value which is close to the result (2.17) obtained without the assumption of small ϵ. Thus formula (4.11) is likely to be a reasonable approximation up to the value 0·5 for ϵ, and is therefore not in itself a source of substantial error. However, it must be admitted that, although our theoretical stability method effectively ignores the occurrence of separation by use of the expansion for small ϵ, the phenomenon has to be faced if comparisons are made with observations for ϵ greater than about 0·3. Clearly, further work is needed on this aspect.

7. Discussion

In this paper we have analysed the stability of the basic flow between rotating eccentric cylinders. The basic velocity has components in the radial and azimuthal directions, but more seriously depends upon the independent co-ordinates ρ and ϕ in these directions, respectively. Thus the straightforward formulation of a linear stability problem leads to *partial* differential equations rather than ordinary differential equations. In our analysis we have two small parameters; δ (or α), the clearance ratio, and ϵ, the eccentricity. As is discussed in § 4, we obtain a tractable problem by considering an appropriate limit as $\delta \rightarrow 0$ and $\epsilon \rightarrow 0$ for fixed T, in such a way that we take account properly of the ϕ variation. Precisely, we assume $\alpha^{\frac{1}{2}} = k\epsilon(2c)^{\frac{1}{2}}$, and thus we obtain an asymptotic result in the limit $\alpha \rightarrow 0$, $\epsilon \rightarrow 0$, with k and T fixed.

The global result (5.21), for the amplitude B as a function of ϕ, arises from an attempt to account rationally for the effect of the variation of the basic flow with the co-ordinate in the flow direction. Of especial interest is the fact that (5.21) is completely unlike any result derivable from a local (or 'parallel-flow') theory. Moreover, the velocity field (5.48)–(5.50) is far more complex than that of the concentric case. However, having taken a first limit $\epsilon \rightarrow 0$, k fixed according to (4.5), we may now consider the effect of k variations.

Two cases are of interest. The first case is $k \rightarrow \infty$, which implies that ϵ is very much smaller than $\delta^{\frac{1}{2}}$ [see (2.14) and (4.5)]. Then we retrieve the concentric case

$$B = B_0(\infty) \tag{7.1}$$

from (5.21), with (5.48)–(5.50) giving the velocity field for the concentric case. In this limit the formula (6.25) for the position of the maximum is

$$\Theta = \tfrac{1}{2}\pi + O(\alpha/2c)^{\frac{1}{2}} + O(\epsilon),$$

since $k\epsilon = (\alpha/2c)^{\frac{1}{2}}$. However, since the velocity field becomes uniform the maximum has no significance. The second case is $k \rightarrow 0$, which implies that $\delta^{\frac{1}{2}}$ is very much smaller than ϵ; then we find from (5.21) that

$$\left.\begin{aligned} B &\rightarrow 0 \quad \text{for} \quad \phi \neq \tfrac{1}{2}\pi, \\ B &\rightarrow B_0(0) \quad \text{for} \quad \phi = \tfrac{1}{2}\pi, \end{aligned}\right\} \tag{7.2}$$

provided we assume that B_0 tends to a constant as $k \to 0$. The result (7.2), in association with (5.48)–(5.50), is certainly not one derivable from local considerations. [We note, however, that only by consideration of the nonlinear problem could we justify the assumption above that $B_0(0)$ is finite; possibilities other than (7.2) may arise and remain to be assessed. Moreover, since (5.48)–(5.50) contain terms $O(k^{-1})$, it may be necessary to require that B_0 is $O(k)$ as $k \to 0$; but again, this needs consideration of the nonlinear terms.]

In the present work it is the property of periodicity in the azimuthal co-ordinate which has enabled us to derive these interesting results for a non-parallel problem. While we do not have the condition of periodicity in boundary-layer instability, the implication is still strong that there may be some similar, as yet undetected, non-local aspects of behaviour in that problem.

One further remark about (5.21) is desirable. Rosenblat & Herbert (1970) have studied the stability of time-dependent basic flows (in thermal convention) using the Galerkin method. Their work and that of this paper are quite independent, but for their problem the model equation (4.1) is far more relevant physically; for if ϕ is replaced by ωt and $R_a V_0(x)\, \partial/\partial\phi$ by $\partial/\partial t$, we have essentially their stability equation for the low frequency case $\omega \to 0$. Also, as can be found by recasting their analysis, a result like (5.21) can be derived. We believe, however, that the present methods have advantages for the algebraically complex stability problem of (3.11)–(3.14).

A final theoretical point that we wish to note is that the present work may be regarded as forming a basis for a nonlinear analysis (analogous to those of Stuart (1958) and Davey (1962)), by which one might calculate the torque and load on a journal bearing, when Taylor vortices are present. However, much more work is needed before any useful remarks can be made on this aspect.

The work of R. C. DiPrima was supported by the Mechanics Branch of the Office of Naval Research and the Army Research Office, Durham. The work of J. T. Stuart was done partly during visits to the Rensselaer Polytechnic Institute in the summers of 1969–1971, with support of the Army Research Office, Durham, and partly at Imperial College. We are deeply indebted to Dr Peter Eagles, who generously performed the computer calculations of Γ, Γ_1 and Γ_2. Also we would like to thank Dr John Vohr for his valuable advice on the interpretation of his experiments on Taylor vortices, and Dr A. Davey for lending us the notes and tables of his 1962 calculations, from which a preliminary value of Γ (23·2) was obtained.

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